# Solution to Math4230 Tutorial 5

1. Let $C$ be a nonempty closed convex set. Prove the following property:

   For any collection of closed convex sets $C_i, i \in I$, where $I$ is an arbitrary index set and $\cap_{i\in I} C_i \neq \emptyset$, we have
$$R_{\cap_{i\in I} C_i} = \cap_{i\in I} R_{C_i}.$$

   **Solution**

   Please refer to the Proof of Proposition 1.4.2 (c). Or you can see the Appendix in the end of this solution.

2. Let $C_1$ and $C_2$ be nonempty convex subsets of $\mathbb{R}^n$ such that $C_2$ is a cone. Suppose that there exists a hyperplane that separates $C_1$ and $C_2$ properly. Show that there exists a hyperplane that separates $C_1$ and $C_2$ properly and passes though the origin.

   **Solution**

   If $C_1$ and $C_2$ can be separated properly, we have from the Proper Separation Theorem that there exists a vector $a \neq 0$ such that
$$\inf_{x\in C_1} a^T x \geq \sup_{x\in C_2} a^T x$$
$$\sup_{x\in C_1} a^T x > \inf_{x\in C_2} a^T x$$
   Let
$$b = \sup_{x\in C_2} a^T x. \tag{1}$$
   And consider the hyperplane
$$H = \{x \mid a^T x = b\},$$
   since $C_2$ is a cone, we have
$$\lambda a^T x = a^T(\lambda x) \leq b < \infty, \quad \forall x \in C_2, \forall \lambda > 0.$$
   This relation implies that $a^T x \leq 0$, for all $x \in C_2$, since otherwise it is possible to choose $\lambda$ large enough and violate that above inequality for some $x \in C_2$. Hence, it follows from (1) that $b \leq 0$. Also by letting $\lambda \to 0$ in the preceding relation, we see that $b \geq 0$. Therefore, we have that $b = 0$ and the hyperplane $H$ contains the origin.

3. Let $C$ be a nonempty closed convex set. Let $W$ be a compact and convex subset of $\mathbb{R}^m$, and let $A$ be an $m \times n$ matrix. The recession cone of the set
$$V = \{x \in C \mid Ax \in W\}$$
   Assuming this set is nonempty. Prove
$$V = R_C \cap N(A),$$
   where $N(A)$ is the nullspace of $A$.

   **Solution** Please refer to the Proof of Proposition 1.4.2 (c). Or you can see the Appendix in the end of this solution.

# Appendix

Proof of Proposition 1.4.2 (c) and (d):

(c) By the definition of direction of recession, $d \in R_{\cap_{i\in I}C_i}$ implies that $x + \alpha d \in \cap_{i\in I}C_i$ for all $x \in \cap_{i\in I}C_i$ and all $\alpha \geq 0$. By Prop. 1.4.1(b), this in turn implies that $d \in R_{C_i}$ for all $i$, so that $R_{\cap_{i\in I}C_i} \subset \cap_{i\in I}R_{C_i}$. Conversely, by the definition of direction of recession, if $d \in \cap_{i\in I}R_{C_i}$ and $x \in \cap_{i\in I}C_i$, we have $x + \alpha d \in \cap_{i\in I}C_i$ for all $\alpha \geq 0$, so $d \in R_{\cap_{i\in I}C_i}$. Thus, $\cap_{i\in I}R_{C_i} \subset R_{\cap_{i\in I}C_i}$.

(d) Consider the closed convex set $\overline{V} = \{x \mid Ax \in W\}$, and choose some $x \in \overline{V}$. Then, by Prop. 1.4.1(b), $d \in R_{\overline{V}}$ if and only if $x + \alpha d \in \overline{V}$ for all $\alpha \geq 0$, or equivalently if and only if $A(x + \alpha d) \in W$ for all $\alpha \geq 0$. Since $Ax \in W$, the last statement is equivalent to $Ad \in R_W$. Thus, $d \in R_{\overline{V}}$ if and only if $Ad \in R_W$. Since $W$ is compact, from part (a) we have $R_W = \{0\}$, so $R_{\overline{V}}$ is equal to $\{d \mid Ad = 0\}$, which is $N(A)$. Since $V = C \cap \overline{V}$, using part (c), we have $R_V = R_C \cap N(A)$. **Q.E.D.**